Stability data, irregular connections and tropical curves

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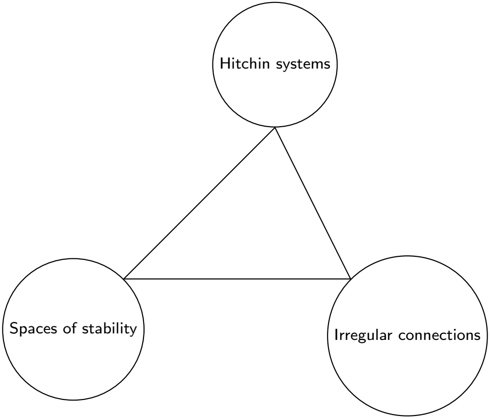
Hitchin systems, stability data and irregular connections



A) Hitchin integrable systems (Higgs bundles)

B) Spaces of stability conditions (à la Bridgeland)

C) Irregular connections on $\mathbb{P}^1$ (after P. Boalch)

Hitchin systems

C Riemann surface, genus $g(C)$.

$SL(2,\mathbb{C})$-**Higgs bundle:** (E,ϕ), E holomorphic vector bundle, $\Lambda^2 E \cong \mathcal{O}_C$, $\phi\colon E \to E \otimes K_C$ holomorphic, $\operatorname{tr}\phi = 0$.

- *Theorem* (Hitchin '86): *the moduli space $\mathcal{M}_{Dol}$ of (E,ϕ)'s is a completely integrable, algebraic, Hamiltonian system.*

Hitchin map: $\mathcal{M}_{Dol} \to B = H^0(K_C^{\otimes 2})\colon [(E,\phi)] \to \det\phi$, with fibres abelian varieties.

- *Observation 1* (Diaconescu-Donagi-Dijkgraaf-Hofman-Pantev '05): *B parameterises a family of non-compact Calabi-Yau threefolds.*

Take $V := K_C^{1/2} \oplus K_C^{1/2}$ and consider $p\colon S^2V \to C$ (rank 3) with the natural map $\det\colon S^2V \subset Hom(V,V^*) \to K_C^{\otimes 2}\colon u \to \det u$. Then,

$$X_b = \{\det = p^*b\} \subset S^2V, \qquad b \in H^0(K_C^{\otimes 2})$$

Locally $v_1 v_2 - y^2 = \det\phi$ (affine conic fibration over C).

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Stability conditions from quadratic differentials

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- *Observation 2* (Smith '13): X_b can be endowed with a symplectic structure, which only depends on $g(C)$: B parameterises Calabi-Yau structures in a fixed symplectic manifold $Y_{g(C)}$.

- *Observation 3* (Thomas '01): the holomorphic volume form on a Calabi-Yau manifold determines a stability condition for Lagrangians.

Aside: Thomas conjectures that L is stable iff it contains a special Lagrangian on its Hamiltonian isotopy class.

Conclusion: The base B of the Hitchin system (actually varying C), should be a '*space of stability conditions*' for Lagrangians in $Y_{g(C)}$.

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Stability conditions ... and quivers

- *Theorem* (Bridgeland-Smith '13): *for suitable irregular Hitchin systems there exists a triangulated category $\mathcal{D}$ with combinatorial description (quivers with potential), such that the base $H^0(K_C^{\otimes 2}(D))$ (varying C) is a connected component of $Stab(\mathcal{D})$.*

Aside: $(\mathcal{A}, Z) \in Stab(\mathcal{D})$ is given by an abelian subcategory $\mathcal{A}$ and an homomorphism $Z\colon K(\mathcal{A}) \to \mathbb{Z}$ such that $Z(K_{>0}(\mathcal{A})) \subset \mathbb{H}$, plus conditions.

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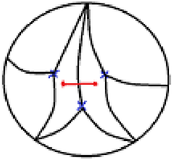
Example: $C = \mathbb{P}^1$, ϕ with singularity of order 4 at ∞, nilpotent leading part at the pole. Generically $\det \phi \in H^0(K_C^{\otimes 2}(7\infty))$, with three simple zeroes. Let R be the path algebra of $\bullet \longrightarrow \bullet$. Then, $\mathcal{A} = Mod^{fin}(R)$ and $\mathcal{D} = D^b(Mod^{fin}(\tilde{R}))$ ($\tilde{R}$ completed (Ginzburg dg) algebra).



What is all this good for?

1) Geometric description of $Stab(\mathcal{D})$ for amenable (combinatorial) $\mathcal{D}$.
2) Combinatorial description of Fukaya.
3) (Conjectural) description of hyperKähler geometry of $\mathcal{M}_{Dol}$!

- *Theorem* (Hitchin '86): *the moduli space $\mathcal{M}_{Dol}$ has a compatible hyperKähler metric g.*
- *Conjecture** (Gaiotto–Moore–Neitzke '09): *the category $\mathcal{D}$ recovers the hyperKähler metric g.*

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Gaiotto–Moore–Neitzke coordinates

- *Observation 4*: the twistor family of holomorphic symplectic structures recovers g.

$$\varpi(\zeta) = -\frac{i}{2\zeta}(\omega_J + i\omega_K) + \omega_I - \frac{i\zeta}{2}(\omega_J - i\omega_K), \qquad \zeta \in \mathbb{C}^* \subset \mathbb{P}^1$$

IDEA (GMN '09): find family $\mathcal{X}_\gamma(\cdot, \zeta)\colon \mathcal{M} \to \mathbb{C}^*$ of holomorphic Darboux coordinates for $\varpi(\zeta)$ solving an integral equation for each $x \in \mathcal{M}_{Dol}$, over $b \in B$:

$$\mathcal{X}_\gamma(x,\zeta) = \mathcal{X}_\gamma^{sf}(x,\zeta) \cdot \\ \cdot \exp\left(\sum_{\gamma' \in K(\mathcal{A}_b)} \Omega(\gamma', b)\langle \gamma, \gamma' \rangle \int_{\mathbb{R}_{<0} Z_b(\gamma')} \frac{d\zeta'}{\zeta'} \frac{\zeta' + \zeta}{\zeta' - \zeta} \log(1 - \mathcal{X}_{\gamma'}(x, \zeta')) \right)$$

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iii) $K(\mathcal{A}_b) \cong H^1(\mathcal{M}_{Dol|b}, \mathbb{Z}) \dashrightarrow$ pairing $\langle \gamma, \gamma' \rangle$, angular coordinates $\theta_{\gamma'}$

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Important Remarks:

A) $\mathcal{X}^{sf}_\gamma$ determine g^{sf} HyperKähler, away from singular fibres,

B) $\mathcal{X}_\gamma$ 'instanton' correct $\mathcal{X}^{sf}_\gamma$ ($\sim$ Gross-Siebert),

C) $\mathcal{X}_\gamma$ discontinuous along $\mathbb{R}_{<0} Z_b(\gamma')$ with $\Omega(\gamma, b) \neq 0$.

D) Kontsevich-Soibelman wall-crossing formula $\Rightarrow$ $\varpi(\zeta) = \sum_{i,j} d\log\mathcal{X}_{\gamma_i}(\zeta) \wedge d\log\mathcal{X}_{\gamma_j}(\zeta)$ smooth

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IDEA (GMN '09): find family $\mathcal{X}_\gamma(\cdot,\zeta)\colon \mathcal{M} \to \mathbb{C}^*$ of holomorphic Darboux coordinates for $\varpi(\zeta)$ solving an integral equation for each $x \in \mathcal{M}_{Dol}$, over $b \in B$:

$$\mathcal{X}_\gamma(x,\zeta) = \mathcal{X}_\gamma^{sf}(x,\zeta)\cdot$$

$$\cdot \exp\left(\sum_{\gamma' \in K(\mathcal{A}_b)} \Omega(\gamma',b)\langle\gamma,\gamma'\rangle \int_{\mathbb{R}_{<0}Z_b(\gamma')} \frac{d\zeta'}{\zeta'}\frac{\zeta'+\zeta}{\zeta'-\zeta}\log(1-\mathcal{X}_{\gamma'}(x,\zeta'))\right)$$

Important Remarks:

A) $\mathcal{X}_\gamma^{sf}$ determine g^{sf} HyperKähler, away from singular fibres,

B) $\mathcal{X}_\gamma$ 'instanton' correct $\mathcal{X}_\gamma^{sf}$ ($\sim$ Gross-Siebert),

C) $\mathcal{X}_\gamma$ discontinuous along $\mathbb{R}_{<0}Z_b(\gamma')$ with $\Omega(\gamma,b) \neq 0$.

D) Kontsevich-Soibelman wall-crossing formula $\Rightarrow$ $\varpi(\zeta) = \sum_{i,j} d\log\mathcal{X}_{\gamma_i}(\zeta) \wedge d\log\mathcal{X}_{\gamma_j}(\zeta)$ smooth

E) $\Omega(\gamma,b) = \Omega(-\gamma,b)$ (symmetric), by Bridgeland axioms.

A very rough approximation of GMN coordinates

$(\Gamma, \langle\cdot,\cdot\rangle)$ a symplectic lattice and $V \subset \Gamma$ a cone. Consider $\mathfrak{g}_V$ the corresponding commutative associative algebra over $\mathbb{C}$

$$e_\gamma * e_{\gamma'} = e_{\gamma+\gamma'}, \qquad \gamma,\gamma' \in V$$

Fix $Z \in \mathrm{Hom}(\Gamma,\mathbb{Z})$. Consider maps $\mathcal{X}\colon \mathbb{C}^* \to \mathrm{Aut}(\widehat{\mathfrak{g}}_V)$ (completion), as unknowns for the integral equation:

$$\mathcal{X}(\zeta)(e_\gamma) = \mathcal{X}^{sf}(\zeta)(e_\gamma) * \\ * \exp_*\left(\sum_{\gamma'\in\Gamma}\Omega(\gamma',Z)\langle\gamma,\gamma'\rangle\int_{\mathbb{R}_{<0}Z(\gamma')}\frac{d\zeta'}{\zeta'}\frac{\zeta'+\zeta}{\zeta'-\zeta}\log_*(1-\mathcal{X}(\zeta')(e_{\gamma'})\right)$$

Remarks:

A) $\mathcal{X}^{sf}(\zeta)(e_\gamma) = \exp_{\mathrm{Aut}(\widehat{\mathfrak{g}}_V)}(\zeta^{-1}Z + \zeta\overline{Z})(e_\gamma) = \exp(\zeta^{-1}Z(\gamma) + \zeta\overline{Z(\gamma)})e_\gamma$

B) $\exp_*, \log_*$ formal series,

C) $\Omega(\gamma, Z) \in \mathbb{Q}, \gamma \in \Gamma$ collection of numbers.

Analogy: $\mathcal{X}(\zeta)$ complexified diffeomorphism of fiber $\mathcal{M}_{Dol|b}$.

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Irregular connections on $\mathbb{P}^1$

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ASSUMPTION: Ω *positive* ($\Omega(\gamma, Z) = 0$ if $\gamma \notin V$).

- *Observation 5* (GMN, ___, Filippini, Stoppa): The integral equation has a solution $\mathcal{X} \colon \mathbb{C}^* \to \mathrm{Aut}(\widehat{\mathfrak{g}}_V)$ by iteration on $\mathcal{X}^{sf}$ (Z-dependent) which defines a connection $\nabla(Z)$ on $\mathbb{P}^1$ with order 2 poles at $\{0, \infty\}$

$$(\partial_\zeta \mathcal{X})\mathcal{X}^{-1}(\zeta) = \left(\zeta^{-2} A_{-1}(Z) + \zeta^{-1} A_0(Z) + A_1(Z)\right) dz \in \Omega^1(\mathrm{Der}(\widehat{\mathfrak{g}}_V))$$

A) Well defined limit $g(Z) = \lim_{\zeta \to 0} \mathcal{X}(\mathcal{X}^{sf})^{-1}(\zeta) \in \mathrm{Aut}(\widehat{\mathfrak{g}}_V)$ (frame)

B) $\Omega(\gamma, Z) \in \mathbb{Q}$ satisfy KS wall-crossing formulae iff $\nabla(Z)$ is isomonodromic.

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Conformal vs large limit

$$g(Z)(e_\gamma) = e_\gamma * \exp_*\langle\gamma, -\sum_T W_T G^0_T(Z)\rangle$$

$$G^0_T(Z) = \int_{\mathbb{R}_{<0}Z(\gamma_T)} \frac{d\zeta'}{\zeta'} \exp(\zeta'^{-1}Z(\gamma_T) + \zeta'\overline{Z(\gamma_T)})e_{\gamma_T} * \prod_{T'} G_{T'}(\zeta', Z)$$

$$G_T(\zeta, Z) = \int_{\mathbb{R}_{<0}Z(\gamma_T)} \frac{d\zeta'}{\zeta'} \frac{\zeta' + \zeta}{\zeta' - \zeta} \exp(\zeta'^{-1}Z(\gamma_T) + \zeta'\overline{Z(\gamma_T)})e_{\gamma_T} * \prod_{T'} G_{T'}(\zeta', Z)$$

$W_T = \Gamma$-valued weight, associated to Γ-decorated rooted tree T.

- *Theorem* (___, Filippini, Stoppa): Consider $R > 0$ a real constant
  - After change of variable $t = R^{-1}\zeta$ there exists a well-defined limit

    $$\lim_{R\to 0} g(RZ)\nabla(RZ)g(RZ)^{-1} = d - \left(\frac{Z}{t^2} + \frac{f(Z)}{t}\right)dt,$$

    where $f(Z)$ is Joyce's holomorphic generating function for DT invariants.
  - As $R \to \infty$, the instanton contribution $G_\bullet(t, RZ)$ as Z crosses a 'wall' in $\mathrm{Hom}(\Gamma, \mathbb{Z})$ encodes counts of tropical curves immersed in $\mathbb{R}^2$.

Conformal vs large limit

$$g(Z)(e_\gamma) = e_\gamma * \exp_*\langle \gamma, -\sum_T W_T G_T^0(Z)\rangle$$

$$G_T^0(Z) = \int_{\mathbb{R}_{<0}Z(\gamma_T)} \frac{d\zeta'}{\zeta'} \exp({\zeta'}^{-1} Z(\gamma_T) + \zeta' \overline{Z(\gamma_T)}) e_{\gamma_T} * \prod_{T'} G_{T'}(\zeta', Z)$$

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Perspectives:

- Symmetric case,
- Link with geometric setup,
- Conformal limit as coordinates in submanifold of opers (Gaiotto),

GRÀCIES!