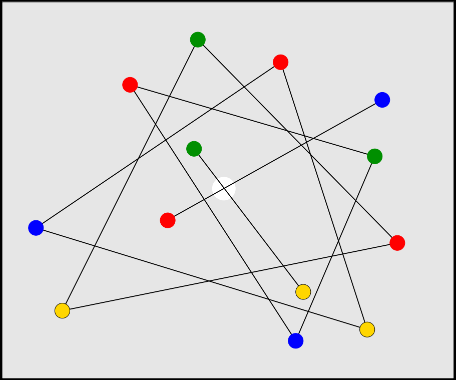
The Story of
"3N Points in a Plane"

Günter M. Ziegler



Plan

0. Why do you care?
1. A short history
2. A short history, with colors
3. A tight colored Tverberg theorem
4. Tverberg strikes back

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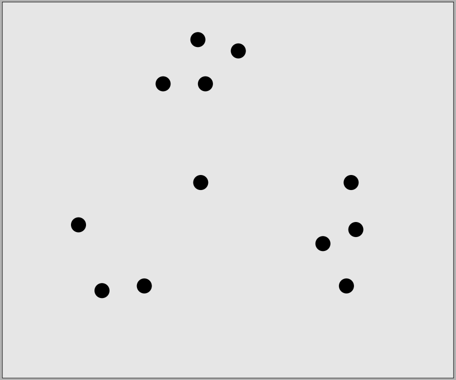
1. A short history

Birch's Theorem 1.

Any 3N points in a plane can be partitioned into N triangles that intersect.



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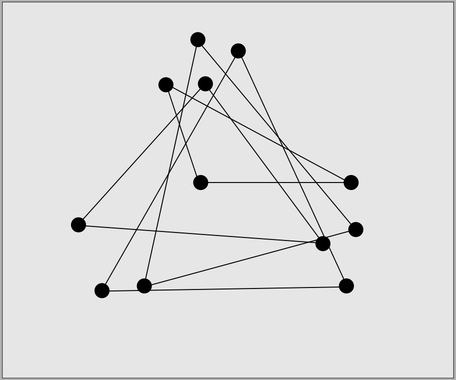

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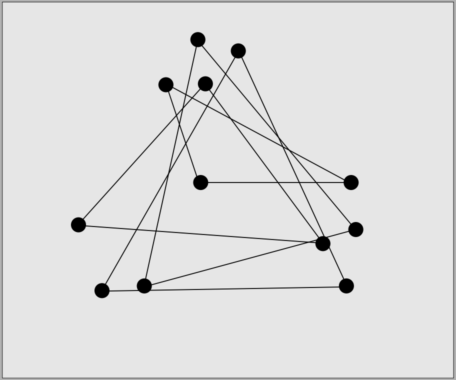

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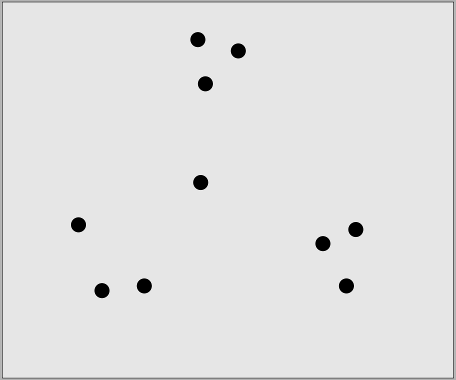
Any $3N-2$ points in a plane can be partitioned into N subsets whose convex hulls intersect.



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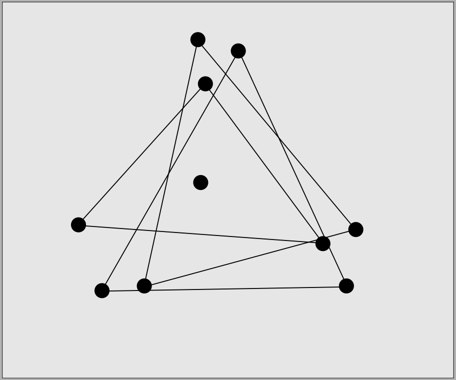

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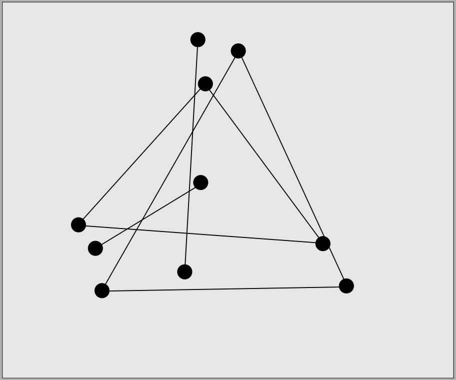

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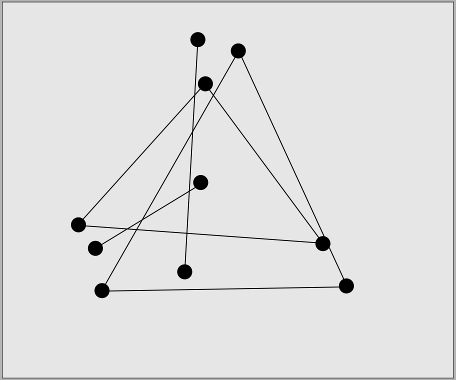
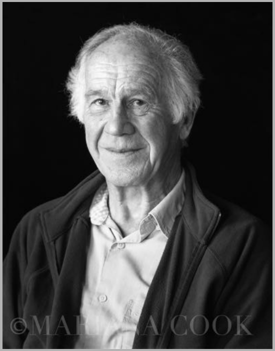


tight!

1. A short history

Bryan John Birch (*1931)

1954 Eggleston course at Cambridge
1956 thesis
1958 Eggleston's book "Convexity"
1959 "On 3N points in a plane"
Math. Proc. Cambridge Phil. Soc.



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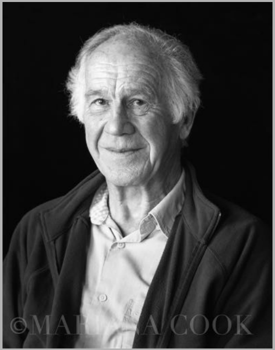

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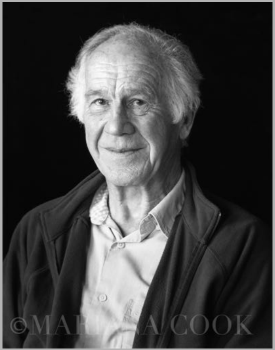

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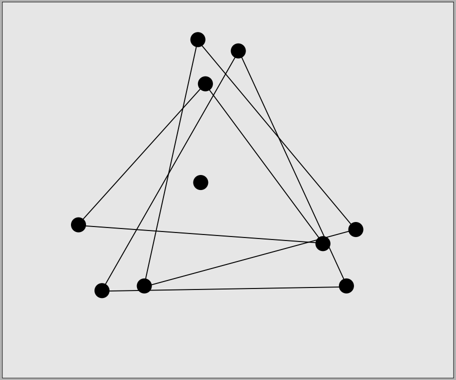
Birch's Conjecture.
$(n+1)N - n$ points in $\mathbb{R}^n$ can be partitioned into N subsets whose convex hulls intersect.



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Change in notation:

$$\begin{array}{lcl} n & \longrightarrow & d \\ N & \longrightarrow & r \\ (n+1)N - n & \longrightarrow & (d+1)r - d \end{array}$$

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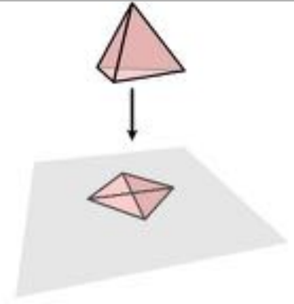
Birch's Conjecture = Tverberg's Theorem.

[Tverberg 1966]

Let $d \geq 1$, $r \geq 2$, $N := (d+1)(r-1)$.
For every affine map

$$f : \Delta_N \longrightarrow \mathbb{R}^d$$

there are r disjoint faces of Δ_N whose f-images intersect.



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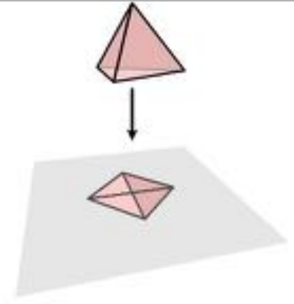

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"Topological Tverberg Theorem".
[Bárány, Shlosman & Szücz 1981]; [Özaydin 1987]; ...)

Let $d \geq 1$, $r \geq 2$, $N := (d+1)(r-1)$.
For every continuous map

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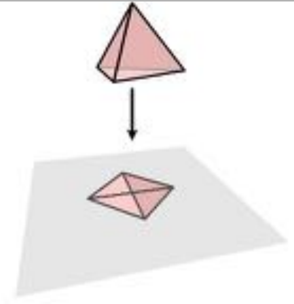

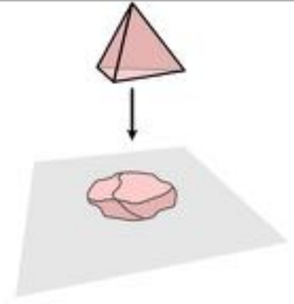

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For other r: open problem. ∎

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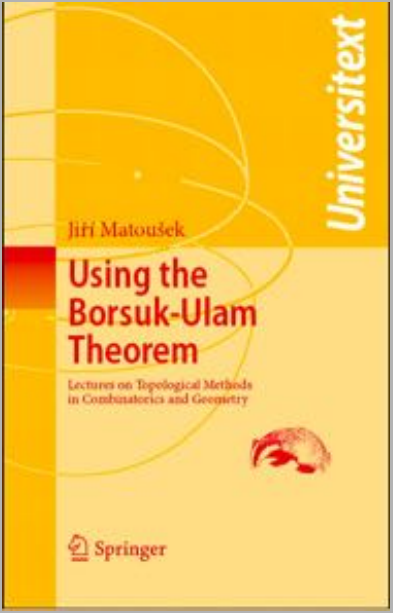

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[Bárány, Füredi & Lovász 1988/1990]:
On the number of halving planes

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Lemma 3. *There is a positive integer t such that the following holds. Assume that $A, B, C \subset \mathbb{R}^2$ are disjoint sets with at least t elements each, such that their union is in general position. Then there exist three disjoint triples $a_i b_i c_i$, $a_i \in A$, $b_i \in B$, $c_i \in C$ $(1 \leq i \leq 3)$ such that $\cap_i \mathrm{conv}(a_i b_i c_i) \neq 0$.*

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The smallest value of t for which we managed to prove this lemma is 4, and we do not have a counterexample even for $t = 3$. For brevity's sake we give the proof for $t = 7$."

2. A short history, with colors

[Bárány & Larman 1991]:

"A colored version of Tverberg's theorem"

Theorem. *Given r red, r white, r green points in the plane, it is possible to form r vertex-disjoint triangles* $\Delta_1, \ldots, \Delta_r$ *in such a way that* Δ_i *has one red, one white, and one green vertex for every* $i = 1, \ldots, r$ *and the intersection of these triangles is non-empty.*

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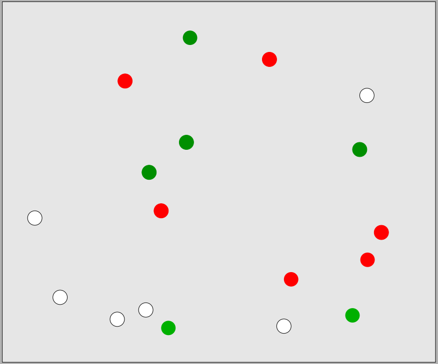

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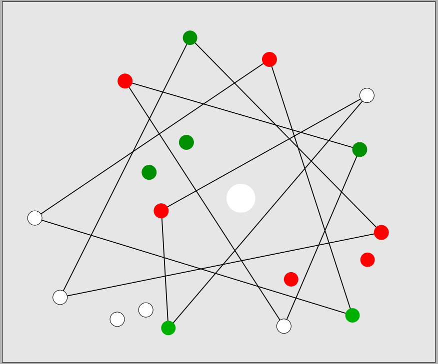

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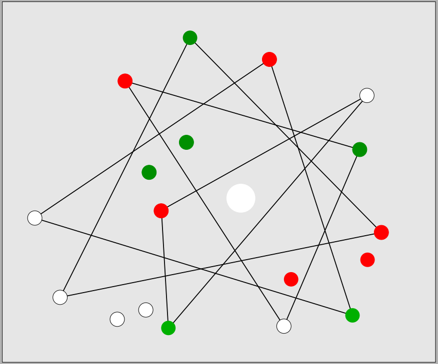

not tight!

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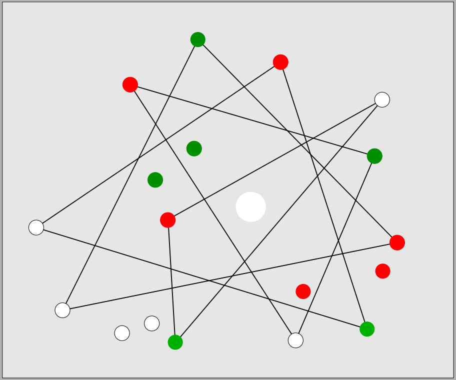
The Colored Tverberg Problem. [Bárány & Larman 1991]

For $d \geq 1$, $r \geq 2$, determine the smallest $N(r,d)$ such that if $N \geq N(r,d)$ the following holds:

If $f : \Delta_N \longrightarrow \mathbb{R}^d$, where the $N+1$ vertices of Δ_N have $d+1$ colors, each color class of size $|C_i| \geq r$, then Δ_N has r disjoint rainbow d-faces whose f-images intersect.



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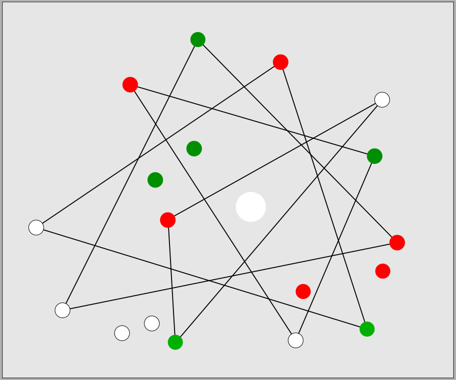

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"Colored Tverberg Theorem".

For $d \geq 1$, $r \geq 2$ suitable, $N \geq t(d+1)-1$, the following holds:

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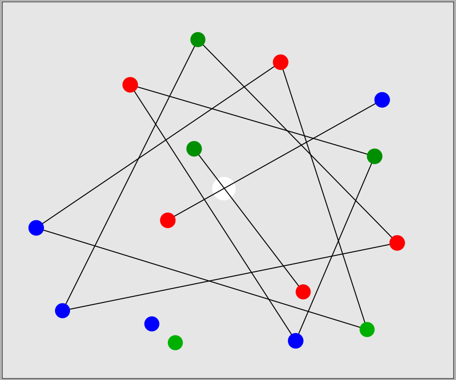

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$t = r$ suffices for $r+1$ prime
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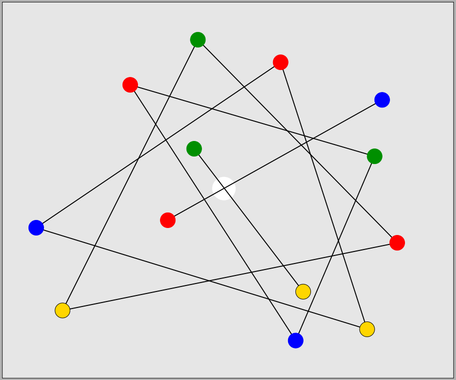
"Tight Colored Tverberg Theorem"
[Blagojević, Matschke & Z. 2009]

Let $d \geq 1$, $r \geq 2$ prime, $N := (d+1)(r-1)$,
$f : \Delta_N \longrightarrow \mathbb{R}^d$ continuous, where the $N+1$ vertices of Δ_N have $m \geq d+2$ colors, each color class of size $|C_i| \leq r-1$.
Then Δ_N has r disjoint rainbow faces whose f-images intersect.



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"Tight Colored Tverberg Theorem"

[Blagojević, Matschke & Z. 2009]

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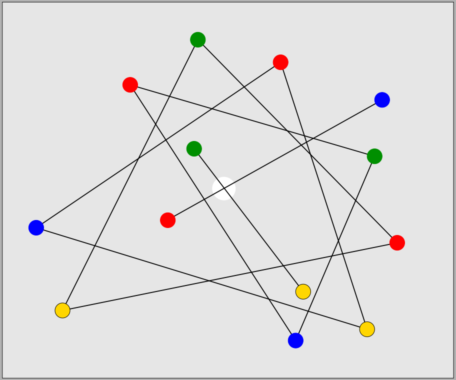
Then Δ_N *has* r *disjoint rainbow faces whose* f*-images intersect.*



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Proof Scheme

Reduction Lemma
([Sarkaria 2000])
It suffices to prove the Theorem for the special case
$|C_0| = |C_1| = \cdots = |C_d| = r - 1, \; |C_{d+1}| = 1$

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Configuration Space/Test Map (CS/TM) Scheme
([Van Kampen 1932], [Sarkaria 1991], [Živaljević 1997+])

- combinatorial configuration space
- deleted joins

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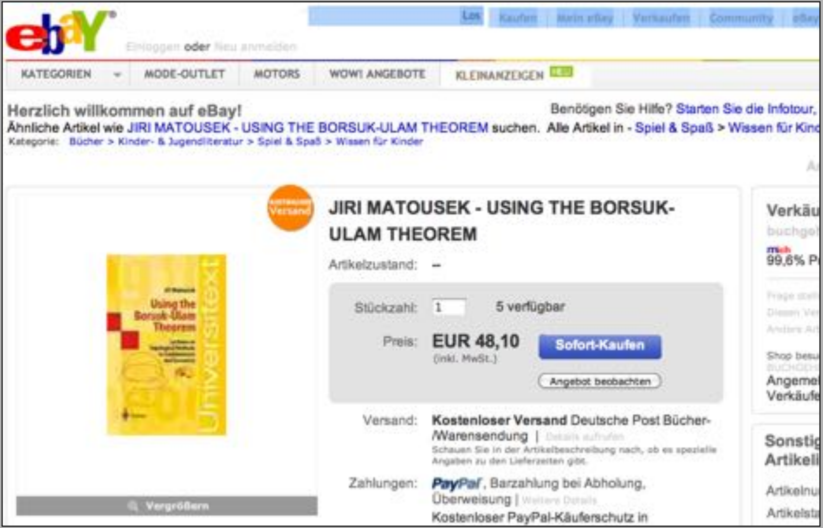
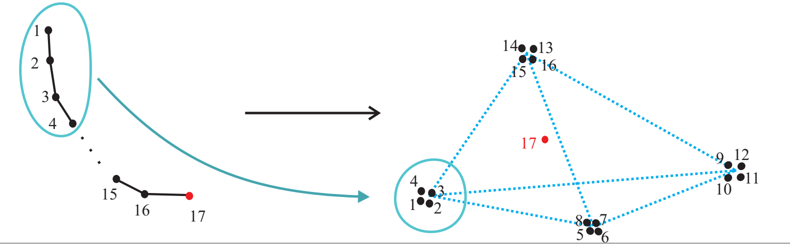


Proof 1 . . . using Degree:

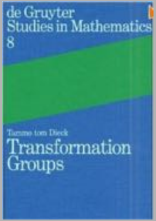
$$F : (\Delta_{r-1,r})^{*(d+1)} \quad \longrightarrow_{\mathbb{Z}_r} \quad S^{N-1}$$

map of orientable pseudomanifolds

- $\deg(F)$ mod r the same for all $\mathbb{Z}_r$-equivariant F
- $\deg(F) = 0$ if F extends to $(\Delta_{r-1,r})^{*(d+1)} * [r]$
- $\deg(F_0) = (r-1)!^d$ for special configuration:



Proof 2 . . . using Obstruction Theory:



[tom Dieck 1987, Sect. II.3]
(apply with care, as $\mathfrak{S}_r$-action not free!)

Proof works, i.e.

$$(\Delta_{r-1,r})^{*(d+1)} * [r] \quad \not\to_{\mathfrak{S}_r} \quad S^{N-1}$$

if and only if

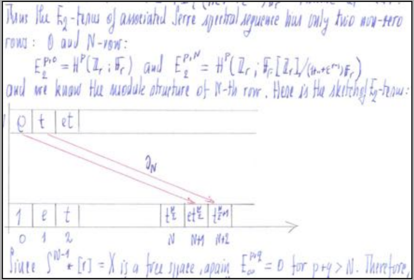
$$r \nmid (r-1)!^{d+1}$$

i.e. if

$$r \text{ is prime.}$$

Proof 3 . . . using Equivariant Index:

[Fadell–Husseini 1988]



- still more complicated
  (. . . equivariant cohomology, index, spectral sequences)
- avoids reduction to the special case
  $|C_0| = |C_1| = \cdots = |C_d| = r - 1, \ \ |C_{d+1}| = 1$
- thus allows for generalizations:
  Tight cases of the Tverberg–Vrećica Conjecture
  [Blagojevic, Matschke & Z. 2011]

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Lemma ("Tverberg unavoidable color class").
[Blagojević, Frick & Z. 2014]
Let $d \geq 1$, $r = p^k$, $N \geq (r-1)(d+1)$,
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Pigeonhole Principle. ■

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Let $d \geq 1$, $r = p^k$, $N \geq 2(r-1)(d+1)$, and let the vertices of Δ_N be in $d+1$ color classes $C_0, \ldots, C_d$ of size $|C_i| \leq 2r-1$.

Then for every continuous $f : \Delta_r \to \mathbb{R}^d$, Δ_N has r disjoint rainbow d-faces whose f-images intersect.

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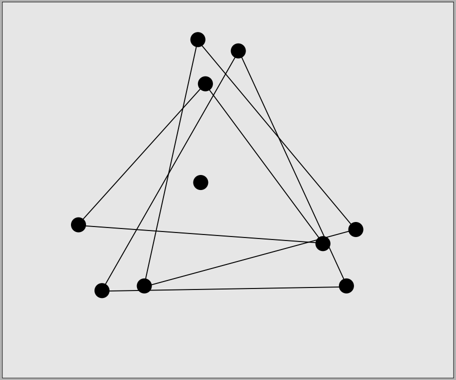

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